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The effects of couple stresses on dislocation strain energy

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Abstract

The correspondence theorem which relates the solutions of displacement boundary value problems in classical and couple stress elasticity is formulated and applied to derive the solutions for edge and screw dislocations in an infinite medium. The effects of couple stresses on the dislocation strain energy is evaluated for both types of dislocations. It is shown that within a radius of influence of each dislocation in a metallic crystal with dislocation density of 10^{10} cm^{-2}, the strain energy contribution from couple stresses (excluding the core energy) is about 15% in the case of an edge dislocation, and about 11% in the case of a screw dislocation. It is then shown that couple stresses make large effect on the total work of tractions acting on the dislocation core surface.

Keywords: Dislocations; Couple stresses; Strain energy; Correspondence theorem; Dislocation core; Hollow dislocation

1. Introduction

In a micropolar continuum the deformation is described by the displacement vector and an independent rotation vector. The rotation vector specifies the orientation of a triad of director vectors attached to each material particle. A particle (material element) can experience a microrotation without undergoing a macrodisplacement. An infinitesimal surface element transmits a force and a couple vector, which give rise to non-symmetric stress and couple stress tensors. The former is related to a non-symmetric strain tensor, and the latter to a non-symmetric curvature tensor, defined as the gradient of the rotation vector. This type of the continuum mechanics was originally introduced by Voigt (1887) and the brothers Cosserat (1909). The fundamentals of the theory were further developed in the sixties, most notably by Günther (1958), Grioli (1960), Aero and Kuvshinskii (1960), Mindlin (1964), and Eringen and Suhubi (1964). In a simplified micropolar theory, the so-called couple stress theory (Toupin, 1962; Mindlin and Tiersten, 1962), the rotation vector is not independent of the displacement vector, but related to it in the same way as in classical continuum mechanics.

The physical rationale for the extension of the classical to micropolar or couple stress theory was that the classical theory was not able to predict the size effect experimentally observed in problems which had a

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geometric length scale comparable to material's microstructural length, such as the grain size in a polycrystalline or granular aggregate. For example, the apparent strength of some materials with stress concentrators such as holes and notches is higher for smaller grain size; for a given volume fraction of dispersed hard particles, the strengthening of metals is greater for smaller particles; the bending and torsional strengths are higher for very thin beams and wires. An extensive list of references to micropolar and couple stress elasticity can be found in review articles by Dhaliwal and Singh (1987) and Jasiuk and Ostoja-Starzewski (1995). The research in couple stress and related non-local and strain-gradient theories of material response (both elastic and plastic) has intensified during the last decade, largely because of an increasing interest to describe the deformation mechanisms and manufacturing of micro- and nanostructured materials and devices, as well as inelastic localization and instability phenomena (Fleck and Hutchinson, 1997; De Borst and Van der Giessen, 1998).

There has been a significant amount of research devoted to dislocation theory in couple stress, micropolar and non-local elasticity. The representative references include Kröner (1963), Mişicu (1965), Teodosiu (1965), Anthony (1970), Knésl and Semela (1972), J.P. Nowacki (1974, 1978), W. Nowacki (1986), Eringen (1977a,b, 1983), Minagawa (1977, 1979), Hsieh et al. (1980), and Gutkin and Aifantis (1996). In this paper we derive the solutions for edge and screw dislocations in an infinite medium by using the correspondence theorem of couple stress elasticity, which relates the solutions of displacement boundary value problems in classical and couple stress elasticity. The basic equations of couple stress elasticity are summarized in Section 2, with an accent given to displacement formulation in Section 3. Both compressible and incompressible elastic materials are considered. The correspondence theorem of couple stress elasticity for the problems with prescribed displacement boundary conditions is formulated in Section 4. The plane strain and anti-plane strain equations of couple stress elasticity are listed in Section 5. The correspondence theorem is applied in Sections 6 and 8 to derive the solutions for edge and screw dislocations in an infinite medium. The solution for the edge dislocation in a hollow cylinder is derived in Section 7. The contribution from couple stresses to dislocation strain energy is evaluated and discussed for both types of dislocations. It is shown that within a radius of influence of each dislocation in a metallic crystal with the dislocation density of 10^{10} cm^{-2}, the strain energy contribution from couple stresses (excluding the core energy) is about 15% in the case of an edge dislocation, and about 11% in the case of a screw dislocation. It is then shown that couple stresses make large effect on the total work of tractions acting on the dislocation core surface. Concluding remarks are given in Section 9.

2. Basic equations of couple stress elasticity

In a micropolar continuum the deformation is described by the displacement vector and an independent rotation vector. In the couple stress theory, the rotation vector φ_i is not independent of the displacement vector u_i but subject to the constraint

$$\varphi_i = \frac{1}{2} e_{ijk}\omega_{jk} = \frac{1}{2} e_{ijk}u_{k,j}, \quad \omega_{ij} = e_{ijk}\varphi_k, \tag{1}$$

as in classical continuum mechanics. The skew-symmetric alternating tensor is e_{ijk}, and ω_{ij} are the rectangular components of the infinitesimal rotation tensor. The latter is related to the displacement gradient and the symmetric strain tensor by $u_{j,i} = \epsilon_{ij} + \omega_{ij}$, where

$$\epsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \quad \omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j}). \tag{2}$$

The comma designates the partial differentiation with respect to Cartesian coordinates x_i.

A surface element dS transmits a force vector T_idS and a couple vector M_idS. The surface forces are in equilibrium with the non-symmetric Cauchy stress t_{ij}, and the surface couples are in equilibrium with the non-symmetric couple stress m_{ij}, such that

$$T_i = n_j t_{ji}, \quad M_i = n_j m_{ji}, \tag{3}$$

where n_j are the components of the unit vector orthogonal to the surface element under consideration. In the absence of body forces and body couples, the differential equations of equilibrium are

$$t_{ji,j} = 0, \quad m_{ji,j} + e_{ijk} t_{jk} = 0. \tag{4}$$

By decomposing the stress tensor into its symmetric and antisymmetric part

$$t_{ij} = \sigma_{ij} + \tau_{ij} \quad (\sigma_{ij} = \sigma_{ji},\ \tau_{ij} = -\tau_{ji}) \tag{5}$$

from the moment equilibrium equation it readily follows that the antisymmetric part can be determined as

$$\tau_{ij} = -\frac{1}{2} e_{ijk} m_{lk,l}. \tag{6}$$

If the gradient of the couple stress vanishes at some point, the stress tensor is symmetric at that point.

The rate of strain energy per unit volume is

$$\dot{W} = \sigma_{ij}\dot{\epsilon}_{ij} + m_{ij}\dot{\kappa}_{ij}, \tag{7}$$

where

$$\kappa_{ij} = \varphi_{j,i} \tag{8}$$

is a non-symmetric curvature tensor. In view of the identity $\omega_{ij,k} = \epsilon_{ki,j} - \epsilon_{kj,i}$, the curvature tensor can also be expressed as

$$\kappa_{ij} = -e_{jkl}\epsilon_{ik,l}. \tag{9}$$

These are the compatibility equations for curvature and strain fields. In addition, there is an identity $\kappa_{ij,k} = \kappa_{kj,i} (= \varphi_{j,ik})$, which defines the compatibility equations for curvature components. The compatibility equations for strain components are the usual Saint Venant's compatibility equations. Since ϵ_{ij} is symmetric and e_{ijk} is skew-symmetric, from Eq. (9) it follows that the curvature tensor in couple stress theory is a deviatoric tensor ($\kappa_{kk} = 0$).

Assuming that the elastic strain energy is a function of the strain and curvature tensors, $W = W(\epsilon_{ij}, \kappa_{ij})$, the differentiation and the comparison with Eq. (7) establishes the constitutive relations of couple stress elasticity

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}}. \tag{10}$$

In the case of isotropic material with the quadratic strain energy,

$$W = \frac{1}{2}\lambda\epsilon_{kk}\epsilon_{ll} + \mu\epsilon_{kl}\epsilon_{kl} + 2\alpha\kappa_{kl}\kappa_{kl} + 2\beta\kappa_{kl}\kappa_{lk}, \tag{11}$$

where μ, λ, α, and β are the Lamé-type constants of isotropic couple stress elasticity. The stress and couple stress tensors are in this case

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}, \quad m_{ij} = 4\alpha\kappa_{ij} + 4\beta\kappa_{ji}. \tag{12}$$

By the positive-definiteness of the strain energy, it follows that $\alpha + \beta > 0$, and $\alpha - \beta > 0$. Thus, α is positive, but not necessarily β. Since the curvature tensor is deviatoric, from the second Eq. (12) it follows that the couple stress is also a deviatoric tensor ($m_{kk} = 0$). In some problems the curvature tensor may be symmetric, and then the couple stress is also symmetric, regardless of the ratio α/β.

If the displacement components are prescribed at a point of the bounding surface of the body, the normal component of the rotation vector at that point cannot be prescribed independently. This implies (e.g., Mindlin and Tiersten, 1962; Koiter, 1964; Germain, 1973) that at any point of a smooth boundary we can specify three reduced stress tractions

$$\overline{T}_i = n_j t_{ji} - \frac{1}{2} e_{ijk} n_j (n_p m_{pq} n_q)_{,k}, \tag{13}$$

and two tangential couple stress tractions

$$\overline{M}_i = n_j m_{ji} - (n_j m_{jk} n_k) n_i. \tag{14}$$

3. Displacement equations of equilibrium

The couple stress gradient can be expressed from Eqs. (9) and (12) as

$$m_{lk,l} = -2\alpha e_{kpq} u_{p,qll}, \tag{15}$$

independently of the material parameter β. The substitution into Eq. (6) gives an expression for the anti-symmetric part of the stress tensor

$$\tau_{ij} = -2\alpha\omega_{ij,kk} = -2\alpha\nabla^2\omega_{ij}, \tag{16}$$

which is also independent of β. The Laplacian operator is $\nabla^2 = \partial^2/\partial x_k \partial x_k$. Consequently, by adding (12) and (16) the total stress tensor is

$$t_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} - 2\alpha\nabla^2\omega_{ij}. \tag{17}$$

Incorporating this into the force equilibrium equations (4), we obtain the equilibrium equations in terms of displacement components

$$\nabla^2 u_i - l^2\nabla^4 u_i + \frac{\partial}{\partial x_i}\left[\frac{1}{1-2\nu}(\mathbf{\nabla}\cdot\mathbf{u}) + l^2\nabla^2(\mathbf{\nabla}\cdot\mathbf{u})\right] = 0, \tag{18}$$

where $\mathbf{\nabla}\cdot\mathbf{u} = u_{k,k}$, the biharmonic operator is $\nabla^4 = \nabla^2\nabla^2$, and

$$l^2 = \frac{\alpha}{\mu}, \quad 1 + \frac{\lambda}{\mu} = \frac{1}{1-2\nu}. \tag{19}$$

The Poisson coefficient is denoted by ν. Upon applying to Eq. (18) the partial derivative $\partial/\partial x_i$, there follows $\nabla^2\epsilon_{kk} = 0$. Thus, the volumetric strain is governed by the same equation as in classical elasticity without couple stresses. The substitution into Eq. (18) yields the final form of the displacement equations of equilibrium

$$\nabla^2 u_i - l^2\nabla^4 u_i + \frac{1}{1-2\nu}\frac{\partial}{\partial x_i}(\mathbf{\nabla}\cdot\mathbf{u}) = 0. \tag{20}$$

Three components of displacement and only two tangential components of rotation may be specified on the boundary. Alternatively, three reduced stress tractions and two tangential couple stress tractions may be specified on a smooth boundary.

The general solution of Eq. (20) can be cast in the form (Mindlin and Tiersten, 1962)

$$u_i = U_i - l^2 \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{U}) - \frac{1}{4(1-\nu)} \frac{\partial}{\partial x_i}[\varphi + \mathbf{x} \cdot (1 - l^2\nabla^2)\mathbf{U}], \tag{21}$$

where the scalar potential φ and the vector potential U_i are solutions of the Laplacian and Helmholtz partial differential equations

$$\nabla^2\varphi = 0, \quad \nabla^2(U_i - l^2\nabla^2 U_i) = 0. \tag{22}$$

The general solution of the latter equation can be obtained by observing that

$$U_i - l^2\nabla^2 U_i = U_i^0 \tag{23}$$

must be a harmonic function, satisfying the Laplace equation $\nabla^2 U_i^0 = 0$. Thus, the general solution can be expressed as $U_i = U_i^0 + U_i^*$, where

$$U_i^* - l^2\nabla^2 U_i^* = 0. \tag{24}$$

3.1. Incompressible materials

For incompressible elastic materials ($\epsilon_{kk} = 0$), the stress response is

$$t_{ij} = 2\mu\epsilon_{ij} - 2\alpha\nabla^2\omega_{ij} - p\delta_{ij}, \tag{25}$$

where $p = p(x_1, x_2, x_3)$ is the pressure field, indeterminate by the constitutive analysis. The corresponding displacement equations of equilibrium are

$$\nabla^2 u_i - l^2\nabla^4 u_i = \frac{1}{\mu}\frac{\partial p}{\partial x_i}. \tag{26}$$

The general solution can be expressed as $u_i = u_i^0 + u_i^*$, where u_i^0 and u_i^* satisfy the non-homogeneous partial differential equations

$$\nabla^2 u_i^0 = \frac{1}{\mu}\frac{\partial p}{\partial x_i}, \tag{27}$$

$$u_i^* - l^2\nabla^2 u_i^* = l^2\frac{1}{\mu}\frac{\partial p}{\partial x_i}. \tag{28}$$

4. The correspondence theorem of couple stress elasticity

For equilibrium problems of couple stress elasticity with prescribed displacement boundary conditions, and with no body forces or body couples present, we state

Theorem. *If $u_i = \hat{u}_i$ is a solution of the Navier equations of elasticity without couple stresses,*

$$\nabla^2\hat{u}_i + \frac{1}{1-2\nu}\frac{\partial}{\partial x_i}(\nabla \cdot \hat{\mathbf{u}}) = 0, \tag{29}$$

then $\hat{u}_i$ is also a solution of differential equations (20) for couple stress elasticity.

Proof. It suffices to prove that $\hat{u}_i$ is a biharmonic function. By applying the Laplacian operator to Eq. (29), we obtain

$$\nabla^4 \hat{u}_i + \frac{1}{1-2\nu} \frac{\partial}{\partial x_i} \nabla^2 (\nabla \cdot \hat{\mathbf{u}}) = 0. \tag{30}$$

Since $\nabla \cdot \hat{\mathbf{u}}$ is a harmonic function, as can be verified from Eq. (29) by applying the partial derivatives $\partial/\partial x_i$, Eq. (30) reduces to

$$\nabla^4 \hat{u}_i = 0. \tag{31}$$

This shows that $\hat{u}_i$ is a biharmonic function, which completes the proof. The correspondence theorem for couple stress elasticity formulated here should be compared with a related principle of association by Sternberg and Muki (1967), and a theorem of correspondence in non-local elasticity by Eringen (1977a,b). □

We now prove that the stress tensor in couple stress elasticity with prescribed displacement boundary conditions and body forces or body couples is a symmetric tensor. From Eq. (29) it readily follows by partial differentiation that the rotation components are harmonic functions ($\nabla^2 \omega_{ij} = 0$, $\nabla^2 \varphi_i = 0$), and substitution into Eq. (16) gives $\tau_{ij} = 0$. In general, the couple stress tensor is still non-symmetric, although in the case of anti-plane strain with prescribed displacement boundary conditions it becomes a symmetric tensor (see Section 5.3).

5. Plane problems of couple stress elasticity

5.1. Plane strain

In plane-strain elasticity the displacement components are $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$, and $u_3 = 0$. The non-vanishing strain, rotation, and curvature components are

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right), \tag{32}$$

$$\varphi_3 = \omega_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), \tag{33}$$

$$\kappa_{13} = \frac{\partial \varphi_3}{\partial x_1}, \quad \kappa_{23} = \frac{\partial \varphi_3}{\partial x_2}. \tag{34}$$

The stress–strain relations are

$$\sigma_{11} = (2\mu + \lambda)\epsilon_{11} + \lambda\epsilon_{22}, \quad \sigma_{22} = (2\mu + \lambda)\epsilon_{22} + \lambda\epsilon_{11}, \tag{35}$$

$$\sigma_{12} = 2\mu\epsilon_{12}, \quad \tau_{12} = -2\alpha\nabla^2\varphi_3. \tag{36}$$

The normal stress $\sigma_{33} = \lambda(\epsilon_{11} + \epsilon_{22})$. The couple stress–curvature relations are

$$m_{13} = 4\alpha\kappa_{13}, \quad m_{31} = 4\beta\kappa_{13}, \quad m_{23} = 4\alpha\kappa_{23}, \quad m_{32} = 4\beta\kappa_{23}. \tag{37}$$

The elastic strain energy per unit volume is

$$W = \frac{1}{2\mu}\left[\sigma_{12}^2 + \frac{1}{2(1+\nu)}(\sigma_{11}^2 + \sigma_{22}^2 - 2\nu\sigma_{11}\sigma_{22} - \sigma_{33}^2)\right] + \frac{1}{8\alpha}(m_{13}^2 + m_{23}^2). \tag{38}$$

Eqs. (32)–(38) can be easily rewritten in terms of polar coordinate components. For example, we have

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \quad \epsilon_{r\theta} = \frac{1}{2r}\left(\frac{\partial u_r}{\partial \theta} + r\frac{\partial u_\theta}{\partial r} - u_\theta\right), \tag{39}$$

$$\varphi_3 = \omega_{r\theta} = \frac{1}{2r}\left[\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right], \quad \kappa_{r3} = \frac{\partial \varphi_3}{\partial r}, \quad \kappa_{\theta 3} = \frac{1}{r}\frac{\partial \varphi_3}{\partial \theta}. \tag{40}$$

Mindlin's stress functions

The rectangular components of stress and couple stress tensors can be expressed in terms of the functions Φ and Ψ as

$$t_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} - \frac{\partial^2 \Psi}{\partial x_1 \partial x_2}, \quad t_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_1 \partial x_2}, \tag{41}$$

$$t_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} - \frac{\partial^2 \Psi}{\partial x_2^2}, \quad t_{21} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} + \frac{\partial^2 \Psi}{\partial x_1^2}, \tag{42}$$

$$m_{13} = \frac{\partial \Psi}{\partial x_1}, \quad m_{23} = \frac{\partial \Psi}{\partial x_2}, \tag{43}$$

where the functions Φ and Ψ satisfy the partial differential equations

$$\nabla^4 \Phi = 0, \quad \nabla^2 \Psi - l^2 \nabla^4 \Psi = 0. \tag{44}$$

The curvature–strain compatibility equations require that the functions Φ and Ψ be related by

$$\frac{\partial}{\partial x_1}(\Psi - l^2 \nabla^2 \Psi) = -2(1-\nu) l^2 \frac{\partial}{\partial x_2}(\nabla^2 \Phi), \tag{45}$$

$$\frac{\partial}{\partial x_2}(\Psi - l^2 \nabla^2 \Psi) = 2(1-\nu) l^2 \frac{\partial}{\partial x_1}(\nabla^2 \Phi). \tag{46}$$

The solution of the equation for Ψ in (44) can be expressed as $\Psi = \Psi^0 + \Psi^*$, where

$$\nabla^2 \Psi^0 = 0, \quad \Psi^* - l^2 \nabla^2 \Psi^* = 0. \tag{47}$$

Thus, Eqs. (45) and (46) can be rewritten as

$$\frac{\partial \Psi_0}{\partial x_1} = -2(1-\nu) l^2 \frac{\partial}{\partial x_2}(\nabla^2 \Phi), \tag{48}$$

$$\frac{\partial \Psi_0}{\partial x_2} = 2(1-\nu) l^2 \frac{\partial}{\partial x_1}(\nabla^2 \Phi). \tag{49}$$

The counterparts of Eqs. (41)–(43), and Eqs. (48) and (49) in polar coordinates are

$$t_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta}, \tag{50}$$

$$t_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta}, \tag{51}$$

$$t_{r\theta} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2}, \tag{52}$$

$$t_{\theta r} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{\partial^2 \Psi}{\partial r^2}, \tag{53}$$

$$m_{r3} = \frac{\partial \Psi}{\partial r}, \quad m_{\theta 3} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \tag{54}$$

and

$$\frac{\partial}{\partial r}(\Psi - l^2\nabla^2\Psi) = -2(1-v)l^2\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2\Phi), \tag{55}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(\Psi - l^2\nabla^2\Psi) = 2(1-v)l^2\frac{\partial}{\partial r}(\nabla^2\Phi). \tag{56}$$

5.2. Anti-plane strain

For the anti-plane strain problems, the displacements are $u_1 = u_2 = 0$, $u_3 = w(x_1, x_2)$. The non-vanishing strain, rotation, and curvature components are

$$\epsilon_{13} = \epsilon_{31} = \frac{1}{2}\frac{\partial w}{\partial x_1}, \quad \epsilon_{23} = \epsilon_{32} = \frac{1}{2}\frac{\partial w}{\partial x_2}, \tag{57}$$

$$\varphi_1 = \omega_{23} = \frac{1}{2}\frac{\partial w}{\partial x_2}, \quad \varphi_2 = \omega_{31} = -\frac{1}{2}\frac{\partial w}{\partial x_1}, \tag{58}$$

$$\kappa_{11} = -\kappa_{22} = \frac{1}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2}, \quad \kappa_{12} = -\frac{1}{2}\frac{\partial^2 w}{\partial x_1^2}, \quad \kappa_{21} = \frac{1}{2}\frac{\partial^2 w}{\partial x_2^2}. \tag{59}$$

It readily follows that

$$t_{13} = \mu\frac{\partial}{\partial x_1}(w - l^2\nabla^2 w), \quad t_{31} = \mu\frac{\partial}{\partial x_1}(w + l^2\nabla^2 w), \tag{60}$$

$$t_{23} = \mu\frac{\partial}{\partial x_2}(w - l^2\nabla^2 w), \quad t_{32} = \mu\frac{\partial}{\partial x_2}(w + l^2\nabla^2 w). \tag{61}$$

The couple stresses are related to the curvature components by

$$m_{11} = 4(\alpha + \beta)\kappa_{11}, \quad m_{22} = 4(\alpha + \beta)\kappa_{22}, \tag{62}$$

$$m_{12} = 4\alpha\kappa_{12} + 4\beta\kappa_{21}, \quad m_{21} = 4\alpha\kappa_{21} + 4\beta\kappa_{12}. \tag{63}$$

Since displacement field is isotropic, the displacement equations of equilibrium (20) reduce to a single equation

$$\nabla^2 w - l^2\nabla^4 w = 0. \tag{64}$$

The general solution can be expressed as $w = w^0 + w^*$, where w^0 and w^* are the solutions of the partial differential equations

$$\nabla^2 w^0 = 0, \quad w^* - l^2\nabla^2 w^* = 0. \tag{65}$$

The non-zero strain, rotation and curvature components in polar coordinates are

$$\epsilon_{\theta 3} = \epsilon_{3\theta} = \frac{1}{2r}\frac{\partial w}{\partial \theta}, \quad \epsilon_{r3} = \epsilon_{3r} = \frac{1}{2}\frac{\partial w}{\partial r}, \tag{66}$$

$$\varphi_r = \omega_{\theta 3} = \frac{1}{2r}\frac{\partial w}{\partial \theta}, \quad \varphi_\theta = \omega_{3r} = -\frac{1}{2}\frac{\partial w}{\partial r}, \tag{67}$$

and

$$\kappa_{rr} = \frac{\partial \varphi_r}{\partial r} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right), \quad \kappa_{r\theta} = \frac{\partial \varphi_\theta}{\partial r} = -\frac{1}{2}\frac{\partial^2 w}{\partial r^2}, \tag{68}$$

$$\kappa_{\theta r} = \frac{1}{r}\frac{\partial \varphi_r}{\partial \theta} - \frac{\varphi_\theta}{r} = \frac{1}{2r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{2r}\frac{\partial w}{\partial r}, \tag{69}$$

$$\kappa_{\theta\theta} = \frac{1}{r}\frac{\partial \varphi_\theta}{\partial \theta} + \frac{\varphi_r}{r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right). \tag{70}$$

The couple stresses are related to the curvature components by

$$m_{rr} = -m_{\theta\theta} = 4(\alpha + \beta)\kappa_{rr}, \tag{71}$$

$$m_{r\theta} = 4\alpha\kappa_{r\theta} + 4\beta\kappa_{\theta r}, \quad m_{\theta r} = 4\alpha\kappa_{\theta r} + 4\beta\kappa_{r\theta}, \tag{72}$$

with the inverse relations

$$\kappa_{r\theta} = \frac{1}{4(\alpha^2 - \beta^2)}(\alpha m_{r\theta} - \beta m_{\theta r}), \quad \kappa_{\theta r} = \frac{1}{4(\alpha^2 - \beta^2)}(\alpha m_{\theta r} - \beta m_{r\theta}). \tag{73}$$

The elastic strain energy per unit volume is

$$W = \frac{1}{2\mu}(\sigma_{r3}^2 + \sigma_{\theta 3}^2) + \frac{1}{4(\alpha + \beta)}\left\{m_{rr}^2 + \frac{1}{2(\alpha - \beta)}[\alpha(m_{r\theta}^2 + m_{\theta r}^2) - 2\beta m_{r\theta}m_{\theta r}]\right\}. \tag{74}$$

5.3. The correspondence theorem for anti-plane strain

For anti-plane strain problems with prescribed displacement boundary conditions, the correspondence theorem of couple stress elasticity reads: If $w = w^0$ is a solution of differential equation of elasticity without couple stresses $\nabla^2 w^0 = 0$, then w^0 is also a solution of differential equations (64) for couple stress elasticity.

The proof is simple. Since w^0 is a harmonic function, it is also a biharmonic function, satisfying Eq. (64). For prescribed displacement boundary conditions, the function w^0 specifies the displacement field in both non-polar and couple stress elasticity.

The stress and couple stress tensors in anti-plane strain problems of couple stress elasticity, in the case of prescribed displacement boundary conditions, are symmetric tensors. Indeed, since the displacement field is a harmonic function, the antisymmetric stress components in Eqs. (60) and (61) vanish, i.e., $\tau_{13} = \tau_{23} = 0$. Thus, the total stress tensor is a symmetric tensor. From Eq. (59) it further follows that the curvature tensor is a symmetric tensor ($\kappa_{12} = \kappa_{21}$). This implies from Eq. (63) that the couple stress tensor is also symmetric ($m_{12} = m_{21}$), regardless of the ratio α/β.

6. Edge dislocation in couple stress elasticity

For the edge dislocation in an infinite medium, the only boundary condition is the displacement discontinuity b, for example imposed along the plane $x_1 > 0$ and $x_2 = 0$. Thus, by the correspondence theorem the displacement field is as in classical elasticity, i.e. (e.g., Hirth and Lothe, 1968),

$$u_1 = \frac{b}{2\pi}\left[\tan^{-1}\frac{x_2}{x_1} + \frac{1}{2(1-\nu)}\frac{x_1x_2}{x_1^2+x_2^2}\right], \tag{75}$$

$$u_2 = -\frac{b}{2\pi}\frac{1}{4(1-\nu)}\left[(1-2\nu)\ln\frac{x_1^2+x_2^2}{b^2} + \frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right]. \tag{76}$$

The stresses are

$$\sigma_{11} = -\frac{\mu b}{2\pi(1-\nu)}\frac{x_2(3x_1^2+x_2^2)}{(x_1^2+x_2^2)^2}, \tag{77}$$

$$\sigma_{22} = \frac{\mu b}{2\pi(1-\nu)}\frac{x_2(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}, \tag{78}$$

$$\sigma_{12} = \frac{\mu b}{2\pi(1-\nu)}\frac{x_1(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}, \tag{79}$$

$$\sigma_{33} = -\frac{\nu\mu b}{\pi(1-\nu)}\frac{x_2}{x_1^2+x_2^2}. \tag{80}$$

The rotation and curvature components are

$$\varphi_3 = -\frac{b}{2\pi}\frac{x_1}{x_1^2+x_2^2}, \tag{81}$$

$$\kappa_{13} = \frac{b}{2\pi}\frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}, \quad \kappa_{23} = \frac{b}{2\pi}\frac{2x_1x_2}{(x_1^2+x_2^2)^2}. \tag{82}$$

The corresponding couple stresses are

$$m_{13} = \frac{2\alpha b}{\pi}\frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}, \quad m_{31} = \frac{2\beta b}{\pi}\frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}, \tag{83}$$

$$m_{23} = \frac{2\alpha b}{\pi}\frac{2x_1x_2}{(x_1^2+x_2^2)^2}, \quad m_{31} = \frac{2\beta b}{\pi}\frac{2x_1x_2}{(x_1^2+x_2^2)^2}. \tag{84}$$

In polar coordinates, the displacements are

$$u_r = \frac{b}{2\pi}\left\{\theta\cos\theta + \frac{1}{4(1-\nu)}\left[1-(1-2\nu)\ln\frac{r^2}{b^2}\right]\sin\theta\right\}, \tag{85}$$

$$u_\theta = -\frac{b}{2\pi}\left\{\theta\sin\theta + \frac{1}{4(1-\nu)}\left[1+(1-2\nu)\ln\frac{r^2}{b^2}\right]\cos\theta\right\}, \tag{86}$$

and the stresses

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{\mu b}{2\pi(1-\nu)}\frac{\sin\theta}{r}, \tag{87}$$

$$\sigma_{r\theta} = \frac{\mu b}{2\pi(1-\nu)}\frac{\cos\theta}{r}, \tag{88}$$

$$\sigma_{33} = -\frac{\nu\mu b}{\pi(1-\nu)}\frac{\sin\theta}{r}. \tag{89}$$

The rotation and curvature components are

$$\varphi_3 = -\frac{b}{2\pi}\frac{\cos\theta}{r}, \quad \kappa_{r3} = \frac{b}{2\pi}\frac{\cos\theta}{r^2}, \quad \kappa_{\theta 3} = \frac{b}{2\pi}\frac{\sin\theta}{r^2}, \tag{90}$$

with the corresponding couple stresses

$$m_{r3} = \frac{2\alpha b}{\pi}\frac{\cos\theta}{r^2}, \quad m_{3r} = \frac{2\beta b}{\pi}\frac{\cos\theta}{r^2}, \tag{91}$$

$$m_{\theta 3} = \frac{2\alpha b}{\pi}\frac{\sin\theta}{r^2}, \quad m_{3\theta} = \frac{2\beta b}{\pi}\frac{\sin\theta}{r^2}. \tag{92}$$

The stress components decay with a distance from the center of dislocation as r^{-1}, while the couple stresses decay as r^{-2}. These also specify the orders of the singularities at the dislocation core when $r \to 0$. The displacement and rotation fields for an edge dislocation in polar elasticity, without the constraint (1), can be found in Nowacki (1986). An analysis allowing for a smooth transition of the displacement field from zero value at the center of the dislocation core to the value b along the cut used to create the dislocation has been done in non-local elasticity by Eringen (1977a). Minagawa (1977, 1979) derived the stress and couple stress fields produced by disclinations and circular dislocations in a micropolar elastic continuum.

6.1. Strain energy

The strain energy (per unit length in x_3 direction) stored within a cylinder bounded by the radii r_0 and R is

$$E = \int_{r_0}^{R}\int_0^{2\pi} W r \,\mathrm{d}r\,\mathrm{d}\theta, \tag{93}$$

where the specific strain energy (per unit volume) is

$$W = \frac{1}{2\mu}[\sigma_{r\theta}^2 + (1-2\nu)\sigma_{rr}^2] + \frac{1}{8\alpha}(m_{r3}^2 + m_{\theta 3}^2) = \frac{\mu b^2}{8\pi^2(1-\nu)^2}\frac{1}{r^2}(1-2\nu\sin^2\theta) + \frac{\alpha b^2}{2\pi^2}\frac{1}{r^4}. \tag{94}$$

Upon the substitution into Eq. (93) and integration, there follows

$$E = \frac{\mu b^2}{4\pi(1-\nu)}\ln\frac{R}{r_0} + \frac{\alpha b^2}{2\pi}\left(\frac{1}{r_0^2} - \frac{1}{R^2}\right). \tag{95}$$

The second term on the right-hand side is the strain energy contribution from the couple stresses. The presence of this term is associated with the work done by the couple stresses on the surfaces $r = r_0$ and $r = R$. This can be seen by writing an alternative expression for the strain energy,

$$E = \frac{1}{2}\int_{r_0}^{R} \sigma_{r\theta}(r,0) b\,\mathrm{d}r + \frac{1}{2}\int_0^{2\pi} M_3\varphi_3 R\,\mathrm{d}\theta - \frac{1}{2}\int_0^{2\pi} M_3\varphi_3 r_0\,\mathrm{d}\theta, \tag{96}$$

with $M_3 = m_{r3}$ given by Eq. (91), and φ_3 given in Eq. (90). The work of the tractions σ_{rr} and $\sigma_{r\theta}$ on the displacements u_r and u_θ over the surface $r = R$ cancels the work of the tractions σ_{rr} and $\sigma_{r\theta}$ over the surface $r = r_0$. These terms are thus not explicitly included in Eq. (96). The second term in Eq. (95) is the strain energy contribution due to last two work terms in Eq. (96). For example, in a metallic crystal with the dislocation density $\rho = 10^{10}$ cm^{-2}, the radius of influence of each dislocation (defined as the average distance between dislocations) is of the order of $\rho^{-1/2} = 100$ nm (Meyers and Chawla, 1999). For an FCC crystal with the lattice parameter $a = 4$ Å and the Burgers vector along the closed packed direction $b = a/\sqrt{2}$, the radius R can be approximately taken as $R = 200b$. By choosing the material length l to be the lattice parameter ($l = \sqrt{2}b$), the couple stress modulus is $\alpha = 2\mu b^2$, and by selecting $r_0 = 2b$ and $\nu = 1/3$, the strain energy contribution from couple stresses in Eq. (95) is 14.5% of the strain energy without couple stresses. The calculations are sensitive to selected value of the dislocation core radius, and larger the value of r_0 smaller the effect of couple stresses in the region beyond r_0. For example, the strain energy contribution from couple stresses in the region between $r_0 = 3b$ and $R = 200b$ decreases to 7%, and in the region between $r_0 = 4b$ and $R = 200b$ to 4.2%. This percentage increases by the decrease of the radius R, and the strain energy contribution from couple stresses in the region between r_0 and $R = 50b$ is 20.7%, 10.5% and 6.6% in the case of $r_0 = 2b$, $3b$ and $4b$, respectively. Such small values of R may be appropriate for problems with extremely high dislocation densities, as arise in localized or non-localized regions of severe plastic deformation (shear bands, wear, wire drawing, high-pressure torsion, equal channel angular pressing), or in the plastically deformed layer behind the shock front (Meyers et al., 2003). In this context, it should be noted that an increase of dislocation density from10^{10} to 10^{11} cm^{-2} results in the decrease of R by the factor of more than 3. It should also be pointed out that the strain energy contribution from couple stresses is likely to be lowered by inclusion of the micropolar effects. The corresponding calculations have not been performed in this paper, but it is known that the effect of couple stresses on stress concentration is less pronounced if the microrotations are assumed to be independent of the displacement field (Kaloni and Ariman, 1967; Cowin, 1970; Lakes, 1985; Eringen, 1999).

6.2. Work of dislocation core tractions

If an edge dislocation is near the free surface or the interface, the contribution from the tractions on the dislocation core surface appears in the final expression for the dislocation strain energy (e.g., Freund, 1994; Lubarda, 1997, 1998). If the dislocation is at the distance from the free surface much greater than the dislocation core radius, it is common practice to evaluate the contribution from the tractions on the dislocation core surface (left after removing the material of the dislocation core) by subjecting the core surface to tractions of the dislocation in an infinite medium, along with the corresponding displacement. This is

$$E_{r_0} = -\frac{1}{2}\int_0^{2\pi} [\sigma_{rr}(r_0,\theta)u_r(r_0,\theta) + \sigma_{r\theta}(r_0,\theta)u_\theta(r_0,\theta)] r_0\,\mathrm{d}\theta - \frac{1}{2}\int_0^{2\pi} m_{r3}(r_0,\theta)\varphi_3(r_0,\theta) r_0\,\mathrm{d}\theta. \tag{97}$$

The first integral in Eq. (97) depends on cut used to create the dislocation (Lubarda, 1998). If the dislocation is created by the displacement discontinuity along the cut at an angle φ (Fig. 1), the evaluation of the above integrals gives

$$E_{r_0} = -\frac{\mu b^2}{8\pi(1-\nu)}\left[\cos 2\varphi - \frac{1}{2(1-\nu)}\right] + \frac{\alpha}{2\pi}\left(\frac{b}{r_0}\right)^2. \tag{98}$$

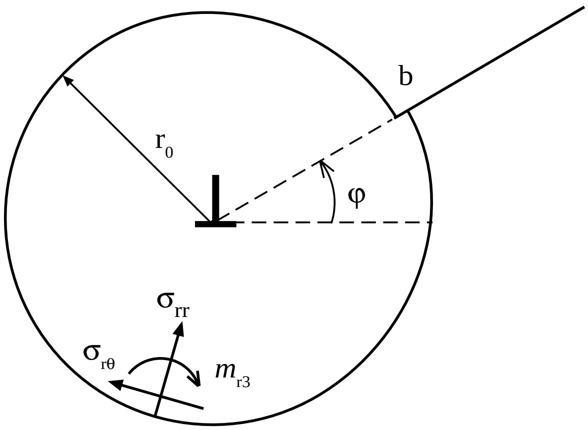

Fig. 1. A material of the dislocation core is removed and its effect on the remaining material represented by the indicated stress and couple stress tractions over the surface $r = r_0$. The slip discontinuity of amount b is imposed along the cut at an angle φ.

Using $\alpha = \mu l^2$, this can be rewritten as

$$E_{r_0} = -\frac{\mu b^2}{2\pi}\left\{\frac{1}{4(1-\nu)}\left[\cos 2\varphi - \frac{1}{2(1-\nu)}\right] - \left(\frac{l}{r_0}\right)^2\right\}. \tag{99}$$

For example, in the case of the displacement discontinuity along the horizontal cut ($\varphi = 0$), the energy becomes

$$E_{r_0} = E^0_{r_0}\left[1 - \frac{8(1-\nu)^2}{1-2\nu}\left(\frac{l}{r_0}\right)^2\right], \tag{100}$$

where

$$E^0_{r_0} = -\frac{1-2\nu}{(1-\nu)^2}\frac{\mu b^2}{16\pi} \tag{101}$$

is the energy contribution without the couple stress effects. For example, if $\nu = 1/3$ and $l = \sqrt{2}b$ (the lattice parameter),

$$E_{r_0} = E^0_{r_0}\left[1 - \frac{64}{3}\left(\frac{b}{r_0}\right)^2\right]. \tag{102}$$

The extraordinary effect of the couple stresses on the ratio $E_{r_0}/E^0_{r_0}$ vs. r_0/b is shown by a solid curve in Fig. 2. If r_0 is equal to $2b$, $3b$ and $4b$, the corresponding ratio $E_{r_0}/E^0_{r_0}$ is equal to −4.33, −1.37 and −0.33, respectively.

If the displacement discontinuity is imposed along the vertical cut ($\varphi = \pi/2$), the energy becomes

$$E_{r_0} = E^0_{r_0}\left[1 + \frac{8(1-\nu)^2}{3-2\nu}\left(\frac{l}{r_0}\right)^2\right], \tag{103}$$

where

$$E^0_{r_0} = \frac{3-2\nu}{(1-\nu)^2}\frac{\mu b^2}{16\pi} \tag{104}$$

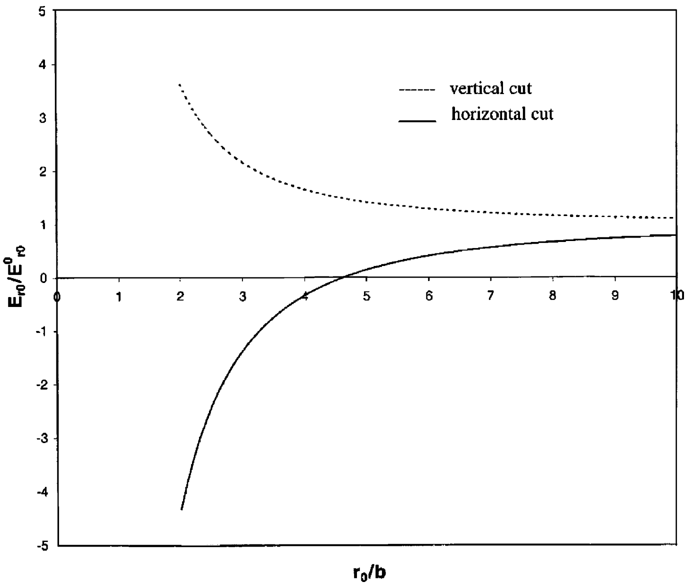

Fig. 2. The ratio of the core surface energies with and without couple stress effects vs. the core radius over the Burgers vector ratio in the case when the edge dislocation is created by the displacement discontinuity along the horizontal cut (solid line) and vertical cut (dashed line). In the former case $E^0_{r_0}$ is defined by Eq. (101), and in the latter case by Eq. (104).

is the energy contribution without the couple stress effects. By setting $\nu = 1/3$ and $l = \sqrt{2}b$, we obtain

$$E_{r_0} = E^0_{r_0}\left[1 + \frac{21}{2}\left(\frac{b}{r_0}\right)^2\right]. \tag{105}$$

The effect of couple stresses on the ratio $E_{r_0}/E^0_{r_0}$ vs. r_0/b is in this case shown by a dashed curve in Fig. 2. If r_0 is equal to $2b$, $3b$ and $4b$, the corresponding ratio $E_{r_0}/E^0_{r_0}$ is equal to 3.62, 2.17 and 1.65, respectively. Although the strain energy associated with the tractions on the surface of the dislocation core is cut dependent, the total strain energy due to dislocation in an infinite or semi-infinite medium is not cut dependent. This was previously discussed in the context of classical elasticity by Lubarda (1997), and an analogous discussion applies for couple stress elasticity.

7. Edge dislocation in a hollow cylinder

The so-called hollow dislocation along the axis of a circular cylinder with inner radius r_0 and the outer radius R is shown in Fig. 3. Both surfaces of the cylinder are required to be stress and couple stress free. The displacement discontinuity of amount b is imposed along the horizontal cut from r_0 to R. The solution is derived from the infinite body solution by superposing an additional solution that cancels the stresses and couple stresses over the inner and outer surface associated with the solution for an edge dislocation in an infinite medium. Thus, we require that the superposed solution satisfies the boundary conditions

$$t_{rr}(R,\theta) = \frac{\mu b}{2\pi(1-\nu)}\frac{\sin\theta}{R}, \quad t_{rr}(r_0,\theta) = \frac{\mu b}{2\pi(1-\nu)}\frac{\sin\theta}{r_0}, \tag{106}$$

$$t_{r\theta}(R,\theta) = -\frac{\mu b}{2\pi(1-\nu)}\frac{\cos\theta}{R}, \quad t_{r\theta}(r_0,\theta) = -\frac{\mu b}{2\pi(1-\nu)}\frac{\cos\theta}{r_0}, \tag{107}$$

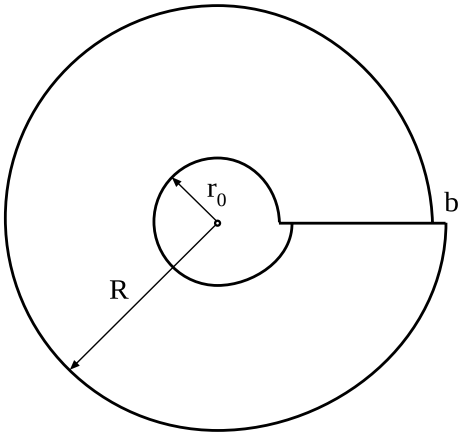

Fig. 3. A slip discontinuity of amount b is imposed along the horizontal cut from the inner radius r_0 to the outer radius R. The inner and outer surface of the cylinder are free from stresses and couple stresses.

$$m_{r3}(R,\theta) = -\frac{2\alpha b}{\pi}\frac{\cos\theta}{R^2}, \quad m_{r3}(r_0,\theta) = -\frac{2\alpha b}{\pi}\frac{\cos\theta}{r_0^2}. \tag{108}$$

This can be accomplished by using the following structure of the Mindlin stress functions:

$$\Phi = \left(A_0 r^3 + B_0\frac{1}{r}\right)\sin\theta, \tag{109}$$

$$\Psi = \left[Ar + B\frac{1}{r} + CI_1\left(\frac{r}{l}\right) + DK_1\left(\frac{r}{l}\right)\right]\cos\theta. \tag{110}$$

From the conditions (55) and (56) it readily follows that

$$A = -16(1-\nu)l^2 A_0, \quad B = 0. \tag{111}$$

The stress and couple stress components of the superposed solution are accordingly

$$t_{rr} = \left[2A_0 r - 2B_0\frac{1}{r^3} + C\frac{1}{rl}I_2\left(\frac{r}{l}\right) - D\frac{1}{rl}K_2\left(\frac{r}{l}\right)\right]\sin\theta, \tag{112}$$

$$t_{r\theta} = -\left[2A_0 r - 2B_0\frac{1}{r^3} + C\frac{1}{rl}I_2\left(\frac{r}{l}\right) - D\frac{1}{rl}K_2\left(\frac{r}{l}\right)\right]\cos\theta, \tag{113}$$

$$m_{r3} = -\left\{16(1-\nu)l^2 A_0 - C\frac{1}{2l}\left[I_0\left(\frac{r}{l}\right) + I_2\left(\frac{r}{l}\right)\right] + D\frac{1}{2l}\left[K_0\left(\frac{r}{l}\right) + K_2\left(\frac{r}{l}\right)\right]\right\}\cos\theta. \tag{114}$$

The expressions for the derivatives of the modified Bessel functions with respect to r/l are used (Watson, 1995, p. 79).

After a lengthy but straightforward derivation it follows that

$$A_0 = \frac{\mu b}{2\pi(1-\nu)}\frac{R^2 - r_0^2}{Ra_1 - r_0 a_2} - \frac{2\alpha b}{\pi}\frac{l}{R^2}\frac{Rb_1 - r_0 b_2}{Ra_1 - r_0 a_2}, \tag{115}$$

$$B_0 = \frac{1}{2} R r_0 \left[\frac{\mu b}{2\pi(1-\nu)} \frac{R a_2 - r_0 a_1}{R a_1 - r_0 a_2} - \frac{2\alpha b}{\pi} \frac{l}{R^2} \frac{a_2 b_1 - a_1 b_2}{R a_1 - r_0 a_2} \right], \tag{116}$$

$$C = 16(1-\nu) l^3 \frac{c_1}{c} A_0 - \frac{2\alpha b}{\pi} \frac{l}{R^2} \frac{c_2}{c}, \tag{117}$$

$$D = -16(1-\nu) l^3 \frac{d_1}{c} A_0 + \frac{2\alpha b}{\pi} \frac{l}{R^2} \frac{d_2}{c}. \tag{118}$$

The introduced parameters are

$$a_1 = 2R^3 - 16(1-\nu) \frac{l^3}{c} (c_1 d_3 - d_1 c_3), \tag{119}$$

$$a_2 = 2r_0^3 - 16(1-\nu) \frac{l^3}{c} (c_1 d_4 - d_1 c_4), \tag{120}$$

$$b_1 = \frac{1}{c} (c_2 d_3 - d_2 c_3), \quad b_2 = \frac{1}{c} (c_2 d_4 - d_2 c_4), \tag{121}$$

where

$$c_1 = \frac{1}{2} \left[K_0\left(\frac{R}{l}\right) + K_2\left(\frac{R}{l}\right) \right] - \frac{1}{2} \left[K_0\left(\frac{r_0}{l}\right) + K_2\left(\frac{r_0}{l}\right) \right], \tag{122}$$

$$c_2 = -\frac{1}{2} \left[K_0\left(\frac{r_0}{l}\right) + K_2\left(\frac{r_0}{l}\right) \right] + \frac{R^2}{2r_0^2} \left[K_0\left(\frac{R}{l}\right) + K_2\left(\frac{R}{l}\right) \right], \tag{123}$$

$$c_3 = \frac{R}{l} K_2\left(\frac{R}{l}\right), \quad c_4 = \frac{r_0}{l} K_2\left(\frac{r_0}{l}\right), \tag{124}$$

and similarly

$$d_1 = -\frac{1}{2} \left[I_0\left(\frac{R}{l}\right) + I_2\left(\frac{R}{l}\right) \right] + \frac{1}{2} \left[I_0\left(\frac{r_0}{l}\right) + I_2\left(\frac{r_0}{l}\right) \right], \tag{125}$$

$$d_2 = \frac{1}{2} \left[I_0\left(\frac{r_0}{l}\right) + I_2\left(\frac{r_0}{l}\right) \right] - \frac{R^2}{2r_0^2} \left[I_0\left(\frac{R}{l}\right) + I_2\left(\frac{R}{l}\right) \right], \tag{126}$$

$$d_3 = -\frac{R}{l} I_2\left(\frac{R}{l}\right), \quad d_4 = -\frac{r_0}{l} I_2\left(\frac{r_0}{l}\right). \tag{127}$$

The parameter c is defined by

$$c = \frac{1}{4} \left[K_0\left(\frac{R}{l}\right) + K_2\left(\frac{R}{l}\right) \right] \left[I_0\left(\frac{r_0}{l}\right) + I_2\left(\frac{r_0}{l}\right) \right] - \frac{1}{4} \left[K_0\left(\frac{r_0}{l}\right) + K_2\left(\frac{r_0}{l}\right) \right] \left[I_0\left(\frac{R}{l}\right) + I_2\left(\frac{R}{l}\right) \right]. \tag{128}$$

The effect of couple stresses on the elastic strain energy as a function of the ratio r_0/b and given R can be evaluated similarly as in previous section. If $R \to \infty$, we obtain the solution for an edge dislocation with a stress free hollow core in an infinite medium, previously considered by Knésl and Semela (1972). In this case $A_0 = A = B = D = 0$, and

$$B_0 = -\frac{\mu b r_0^2}{4\pi(1-\nu)} - \frac{2\alpha b}{\pi}\frac{K_2\left(\frac{r_0}{l}\right)}{K_0\left(\frac{r_0}{l}\right)+K_2\left(\frac{r_0}{l}\right)}, \tag{129}$$

$$D = \frac{4\alpha b l}{\pi r_0^2}\frac{1}{K_0\left(\frac{r_0}{l}\right)+K_2\left(\frac{r_0}{l}\right)}. \tag{130}$$

If couple stresses are neglected, Eqs. (115)–(118) yield

$$A_0 = \frac{\mu b}{4\pi(1-\nu)}\frac{1}{R^2+r_0^2}, \quad B_0 = -R^2 r_0^2 A_0, \quad A = C = D = 0. \tag{131}$$

The corresponding stresses are

$$\sigma_{rr} = -\frac{\mu b}{2\pi(1-\nu)}\left[\frac{1}{r} - \frac{1}{R^2+r_0^2}\left(r + \frac{R^2 r_0^2}{r^3}\right)\right]\sin\theta, \tag{132}$$

$$\sigma_{r\theta} = \frac{\mu b}{2\pi(1-\nu)}\left[\frac{1}{r} - \frac{1}{R^2+r_0^2}\left(r + \frac{R^2 r_0^2}{r^3}\right)\right]\cos\theta, \tag{133}$$

$$\sigma_{\theta\theta} = -\frac{\mu b}{2\pi(1-\nu)}\left[\frac{1}{r} - \frac{1}{R^2+r_0^2}\left(3r - \frac{R^2 r_0^2}{r^3}\right)\right]\sin\theta, \tag{134}$$

in agreement with the solution for the Volterra edge dislocation from classical elasticity (Love, 1944). The results should also be compared with those presented by Hirth and Lothe (1968, p. 77), who remove the tractions on the inner and outer surface of the cylinder only to within the first order terms in r_0/R.

8. Screw dislocation in couple stress elasticity

The displacement field for a screw dislocation with imposed displacement discontinuity b along the plane $x_1 > 0$ and $x_2 = 0$ is as in classical elasticity

$$w = \frac{b}{2\pi}\tan^{-1}\frac{x_2}{x_1} = \frac{b}{2\pi}\theta. \tag{135}$$

This follows by a correspondence theorem since only displacement boundary conditions are prescribed. The stresses and couple stresses associated with (135) are

$$\sigma_{13} = -\frac{\mu b}{2\pi}\frac{x_2}{x_1^2+x_2^2}, \quad \sigma_{23} = \frac{\mu b}{2\pi}\frac{x_1}{x_1^2+x_2^2}, \tag{136}$$

$$m_{11} = -m_{22} = -\frac{(\alpha+\beta)b}{\pi}\frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}, \tag{137}$$

$$m_{12} = m_{21} = -\frac{(\alpha+\beta)b}{\pi}\frac{2x_1x_2}{(x_1^2+x_2^2)^2}. \tag{138}$$

The components of the stress and couple stress tensors along the polar directions are

$$\sigma_{\theta 3} = \frac{\mu b}{2\pi}\frac{1}{r}, \quad \sigma_{r3} = 0, \tag{139}$$

$$m_{rr} = -m_{\theta\theta} = -\frac{(\alpha+\beta)b}{\pi}\frac{1}{r^2}, \quad m_{r\theta} = m_{\theta r} = 0. \tag{140}$$

The stress components decay with a distance from the center of dislocation as r^{-1}, while the couple stresses decay as r^{-2}. These also specify the orders of the singularities at the dislocation core when $r \to 0$. The displacement and rotation fields for a screw dislocation in micropolar elasticity can be found in Nowacki (1986), although his statement (at the bottom of page 327) that in classical theory of dislocations we have $\varphi_1 = \varphi_2 = 0$ is apparently a misprint, since the rotation components from Eq. (135) do not vanish but are equal to

$$\varphi_1 = \frac{1}{2}\frac{\partial w}{\partial x_2} = \frac{b}{4\pi}\frac{x_1}{x_1^2 + x_2^2}, \quad \varphi_2 = -\frac{1}{2}\frac{\partial w}{\partial x_1} = \frac{b}{4\pi}\frac{x_2}{x_1^2 + x_2^2}. \tag{141}$$

The analysis allowing for a smooth transition of the displacement field from zero value at the center of the dislocation core to the value b along the cut used to create the dislocation has been done in non-local elasticity by Eringen (1977b, 1983), and in gradient elasticity by Gutkin and Aifantis (1996).

8.1. Strain energy

The strain energy (per unit length in x_3 direction) stored within a cylinder bounded by the radii r_0 and R is

$$E = \int_{r_0}^{R} W 2\pi r \, \mathrm{d}r, \tag{142}$$

where the specific strain energy (per unit volume) is

$$W = \frac{1}{2\mu}\sigma_{\theta 3}^2 + \frac{1}{4(\alpha+\beta)}m_{rr}^2 = \frac{\mu b^2}{8\pi^2}\frac{1}{r^2} + \frac{(\alpha+\beta)b^2}{4\pi^2}\frac{1}{r^4}. \tag{143}$$

Upon the substitution into Eq. (142) and integration, there follows

$$E = \frac{\mu b^2}{4\pi}\ln\frac{R}{r_0} + \frac{(\alpha+\beta)b^2}{4\pi}\left(\frac{1}{r_0^2} - \frac{1}{R^2}\right). \tag{144}$$

The second term on the right-hand side is the strain energy contribution from the couple stresses. The presence of this term is associated with the work done by the couple stresses on the surfaces $r = r_0$ and $r = R$. This can be seen by writing an alternative expression for the strain energy,

$$E = \frac{1}{2}\int_{r_0}^{R} \sigma_{\theta 3}(r,0) b \, \mathrm{d}r + \frac{1}{2}\int_0^{2\pi} M_r \varphi_r R \, \mathrm{d}\theta - \frac{1}{2}\int_0^{2\pi} M_r \varphi_r r_0 \, \mathrm{d}\theta, \tag{145}$$

with $M_r = m_{rr}$ given by Eq. (140), and

$$\varphi_r = \omega_{\theta z} = \frac{1}{2r}\frac{\partial w}{\partial \theta} = \frac{1}{4\pi}\frac{b}{r} \tag{146}$$

being the r component of the rotation vector. Since $\sigma_{r3} = 0$, there is no work of stress traction on the displacement w over the surfaces $r = r_0$ and $r = R$. The second term in Eq. (144) is the strain energy contribution due to last two work terms in Eq. (145). For example, if we set $R = 200b$, $r_0 = 2b$, and $\alpha + \beta = 2\mu b^2$, the energy contribution from couple stresses in Eq. (144) is 10.9% of the strain energy without couple stresses. In the classical elasticity a cylindrical surface around the screw dislocation at its center is stress free. On the other hand, the solution derived in this section is characterized by the presence of the constant couple stress m_{rr} along that surface. However, since m_{rr} in Eq. (140) does not depend on θ, the reduced traction $\bar{t}_{r3}$ vanishes on the cylindrical surface $r = \text{const}$.

9. Conclusions

We have derived in this paper the solutions for edge and screw dislocations in an infinite medium by using the correspondence theorem of couple stress elasticity, which relates the solutions of displacement boundary value problems in classical and couple stress elasticity. The contribution from couple stresses to dislocation strain energy is evaluated and discussed for both types of dislocations. It is shown that within a radius of influence of each dislocation in a metallic crystal with the dislocation density of 10^{10} cm^{-2}, the strain energy contribution from couple stresses is about 15% in the case of an edge dislocation, and about 11% in the case of a screw dislocation (excluding the energy of the dislocation core of radius $r_0 = 2b$). This contribution decreases with an increasing size of the dislocation core. For example, the strain energy contribution from couple stresses for an edge dislocation in the region between $r_0 = 3b$ and $R = 200b$ decreases to 7%, and in the region between $r_0 = 4b$ and $R = 200b$ to 4.2%. This percentage increases by the decrease of the radius R, and the strain energy contribution from couple stresses in the region between r_0 and $R = 50b$ is 20.7%, 10.5% and 6.6% in the case of $r_0 = 2b, 3b$ and $4b$, respectively. Such small values of R may be appropriate for problems with extremely high dislocation densities within the localized or non-localized regions of severe plastic deformation. It is then shown that couple stresses make large effect on the total work of tractions acting on the dislocation core surface. The solution for the edge dislocation in a hollow cylinder (Volterra dislocation) in the presence of couple stresses is also derived. The extension of the present work is in progress to incorporate the effect of couple stresses on the interaction forces between dislocations on parallel and intersecting slip planes, and the dislocation interactions with straight and curved free surfaces or rigid boundaries. For example, for an edge dislocation near the stress-free straight boundary the image dislocation cancels both the normal and couple stress components at the boundary, so that only the shear stress component has to be removed by superposition of an auxiliary problem to achieve the stress-free boundary condition. A study of the organized dislocation structures in couple stress elasticity, such as that presented by Lubarda et al. (1993) and Lubarda and Kouris (1996a,b) in the case of classical elasticity, may also be of interest. In addition, the incorporation of couple stresses in the analysis of strain relaxation in thin films (Freund, 1994; Lubarda, 1998) is worthwhile. Since couple stresses significantly affect dislocation strain energies, they may also have a significant effect on the critical film thickness and the conditions for the formation of interface dislocation arrays. The solutions for eigenstrain and inhomogeneity problems for circular inclusions in anti-plane strain couple stress elasticity are presented in the accompanying paper (Lubarda, 2003).

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